Int. J. Heat Mass Transfer. Vol. 6, pp. 833–840. Pergamon Press 1963. Printed in Great Britain.

NOTES ON THE TRANSFER AT THE INTERFACE OF TWO INDEPENDENTLY STIRRED LIQUIDS

J. SZEKELY

Department of Metallurgy, Royal School of Mines, Imperial College, London

(*Received* 29 *January* 1963)

Abstract—A model is proposed for describing transfer across the interface of two independently stirred liquids. Independent stirring was considered to effect different periods of surface renewal in the two phases. A scheme is proposed for evaluating a time averaged value of the flux crossing the interface, and an analytical solution is obtained for the simplest case of nonequal surface renewal times, i.e. when the surface renewal time in one phase is twice that in the other.

It is shown, that even for this simple case, the overall resistance to transfer can no longer be described as "composed of two linearly additive terms each containing parameters relating to only one of the phases". The simplest schematic representation would be: overall resistance = resistance in one phase + resistance in the other phase + an interaction parameter.

NOMENCLATURE

$A_1(x)$, $A_2(x)$,	particular integrals of equations (9A) and (10A) respectively;
C_1, C_2, C_3, C_4,	integration constants;
$D = \dfrac{\mathrm{d}}{\mathrm{d}x}$,	differential operator;
k_1, k_2,	thermal conductivities in phases 1 and 2 respectively;
l, m, n,	integers;
$M = \dfrac{k_1\kappa_1^{-\frac{1}{2}}\, k_2^{-1}}{k_1\kappa_1^{-\frac{1}{2}} + k_2\kappa_2^{-\frac{1}{2}}}(\mathscr{T}_1 - \mathscr{T}_2)$;	
$N = \dfrac{1}{4k_2t_1}$;	
p,	see definition of $\bar{T}_1$ and $\bar{T}_2$;
$q = \dfrac{p}{\kappa}$;	
$Q(t)$,	instantaneous flux crossing the interface;
Q_τ,	time averaged value of flux for period τ;
$\mathscr{T}_1, \mathscr{T}_2$,	uniform bulk temperatures in phases 1 and 2 respectively;
T_1, T_2,	instantaneous temperatures in phases 1 and 2;
$\mathscr{L}(T_1) = \bar{T}_1 = \int_0^\infty e^{-pt} \,.\, T_1\,\mathrm{d}t$;	
t,	time;
t_1, t_2,	time intervals;
x,	distance from the interface;
κ_1, κ_2,	thermal diffusivities in phases 1 and 2.

INTRODUCTION

In a previous paper [1] a model was proposed for the transfer of heat or mass at the bubble-stirred interface of two immiscible liquids. Two immiscible liquids phases (1) and (2) were considered, extending from $x = 0_{(+)}$ to $x = \infty$ and $x = 0_{(-)}$ to $x = -\infty$ respectively. Values were assigned to the respective bulk temperatures $\mathscr{T}_1$, $\mathscr{T}_2$ thermal diffusivities, κ_1, κ_2 and thermal conductivities k_1, k_2 of the two phases. Finally, the instantaneous temperatures near the surface were given as T_1 and T_2 respectively.

It was considered that when a bubble crossed the interface it destroyed instantaneously the temperature gradients on both sides of the $x = 0$ plane. Thus at this instant $T_1 = \mathscr{T}_1$ for $x > 0$ and $T_2 = \mathscr{T}_2$ for $x < 0$. Then there exists unsteady state conduction from phase 1 to phase 2 with a corresponding re-establishment of temperature gradients until the arrival of the next bubble when the whole cycle starts again. It was shown that such a system can be represented by the two equations:

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\kappa_1}\frac{\partial T_1}{\partial t}, \quad \text{for } x > 0 \qquad (1)$$

and

$$\frac{\partial^2 T_2}{\partial x^2} = \frac{1}{\kappa_2}\frac{\partial T_2}{\partial t}, \quad \text{for } x < 0 \qquad (2)$$

connected by the following boundary conditions.

$$T_1 = T_2 \qquad \text{at } x = 0,\ t > 0 \qquad (3)$$

$$k_1 \frac{\partial T_1}{\partial x} = k_2 \frac{\partial T_2}{\partial x} \quad \text{at } x = 0,\ t > 0 \qquad (4)$$

$$T_1 = \mathscr{T}_1 \qquad \text{at } t = 0, \text{ for } x > 0 \qquad (5)$$

$$T_2 = \mathscr{T}_2 \qquad \text{at } t = 0, \text{ for } x < 0 \qquad (6)$$

$$T_1 = \mathscr{T}_1 \qquad \text{at } x = \infty, \text{ for } t \geqslant 0 \qquad (7)$$

$$T_2 = \mathscr{T}_2 \qquad \text{at } x = -\infty, \text{ for } t \geqslant 0. \qquad (8)$$

The solution of equations (1) and (2) was quoted from Carslaw and Jaeger [2] as

$$T_1 - \mathscr{T}_2 = \frac{k_1\kappa_1^{-\frac{1}{2}}}{k_2\kappa_2^{-\frac{1}{2}} + k_1\kappa_1^{-\frac{1}{2}}} \times (\mathscr{T}_1 - \mathscr{T}_2) \left[1 + \frac{k_2\kappa_2^{-\frac{1}{2}}}{k_1\kappa_1^{-\frac{1}{2}}} \operatorname{erf} \frac{x}{2\sqrt{(\kappa_1 t)}}\right] \qquad (9)$$

and

$$T_2 - \mathscr{T}_2 = \frac{k_1\kappa_1^{-\frac{1}{2}}}{k_1\kappa_1^{-\frac{1}{2}} + k_2\kappa_2^{-\frac{1}{2}}} \times (\mathscr{T}_1 - \mathscr{T}_2) \operatorname{erfc} \frac{|x|}{2\sqrt{(\kappa_2 t)}}. \qquad (10)$$

Expressions were also derived for the instantaneous flux $[Q(t)]$

$$Q(t) = k_1 \left(\frac{\partial T_1}{\partial x}\right)_{x=0} = k_2 \left(\frac{\partial T_2}{\partial x}\right)_{x=0} \qquad (11)$$

and time averaged flux Q_{t_e} for the period t_e, i.e. the time interval between the arrival of two successive bubbles.

$$Q_{t_e} = \frac{1}{t_e}\int_0^{t_e} Q(t)\,\mathrm{d}t \qquad (12)$$

and finally the overall heat-transfer coefficient was given as

$$h = \frac{Q_{t_e}}{\mathscr{T}_1 - \mathscr{T}_2} = \frac{2k_1}{\sqrt{(\pi\kappa_1 t_e)}} \times \frac{k_2\kappa_2^{-\frac{1}{2}}}{k_1\kappa_1^{-\frac{1}{2}} + k_2\kappa_2^{-\frac{1}{2}}} \qquad (13)$$

rearranging equation (11) we have

$$h = \frac{2}{\sqrt{(\pi)}} \frac{1}{\dfrac{\sqrt{(t_e)}}{k_1\kappa_1^{-\frac{1}{2}}} + \dfrac{\sqrt{(t_e)}}{k_2\kappa_2^{-\frac{1}{2}}}} \qquad (14)$$

i.e. the overall resistance to transfer is composed of two linearly additive terms each containing parameters relating to only one of the phases. In other words: overall resistance = resistance in phase 1 + resistance in phase 2.

MODEL FOR INDEPENDENT STIRRING

It is of interest to consider the case when the two phases are stirred independently.

Similarly to the problem involving the bubble-stirred interface, consider two immiscible liquid phases (1) and (2) extending from $x = 0_{(+)}$ to $x = \infty$ and $x = 0_{(-)}$ to $x = -\infty$ respectively. Assign values to the bulk temperatures $\mathscr{T}_1$, $\mathscr{T}_2$ thermal conductivities k_1, k_2, thermal diffusivities κ_1, κ_2 where the subscripts 1 and 2 refer to the appropriate phases. Consider an idealized case when due to stirring at different rates, the surface (this is deemed to include all material in the vicinity of the surface where the temperature is not equal to that of the bulk) of phase 1 is renewed (i.e. replaced with material from the bulk) at time intervals $t_1, 2t_1, 3t_1 \ldots nt_1$ and that of phase 2 at intervals $t_2, 2t_2, 3t_2 \ldots mt_2$.*

Consider a cycle of length τ, starting at a time $t = 0$ when the renewal of surfaces coincides in both phases, extending to a time $t = \tau$ when the same situation is reproduced. It is evident that τ must be the smallest common multiple of t_1 and t_2.

Thus at $t = 0$ we have: (Fig. 1)

$$T_1 = \mathscr{T}_1 \qquad \text{for } x > 0 \qquad (5)$$

and

$$T_2 = \mathscr{T}_2 \qquad \text{for } x < 0. \qquad (6)$$

At $0 < t < t_1$ we have (Fig. 2)

$$\left.\begin{aligned} T_1 &= \phi_1(x, t)_1 \\ T_2 &= \phi_2(x, t)_1 \end{aligned}\right\} \qquad (15)$$

---

* It is noted that an additional assumption is implicit in the above formulation, namely that it is *possible* to renew the surface of one phase without affecting the other phase, i.e. the existence of "perfect slip" at the interface.

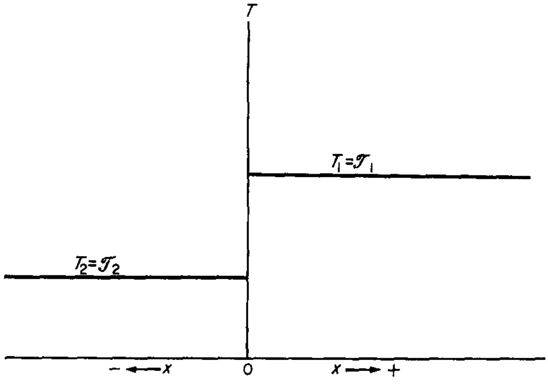

FIG. 1.

At $t = t_1$ (Fig. 3)

$$\left.\begin{aligned} T_1 &= \mathscr{T}_1 \\ T_2 &= \phi_2(x, t)_1 \end{aligned}\right\} \quad (16)$$

If $1 < (t_2/t_1) < 2$; at $t = t_2$ (Fig. 4) (17)

$$\left.\begin{aligned} T_1 &= \phi_1(x, t)_2 \\ T_2 &= \mathscr{T}_2. \end{aligned}\right\} \quad (18)$$

At $t = t_{2(+)}$ we have

$$\left.\begin{aligned} T_1 &= \phi_1(x, t)_3 \\ T_2 &= \phi_2(x, t)_2 \end{aligned}\right\} \quad (19)$$

and in general, for $0 < t < \tau$, $t \neq lt_1$, $t \neq lt_2$ where $lt = 1, 2 \ldots$

$$\left.\begin{aligned} T_1 &= \phi_1(x, t)_n \\ T_2 &= \phi_2(x, t)_m. \end{aligned}\right\} \quad (20)$$

The problem is to evaluate point values and time averaged values of the flux crossing the $x = 0$ plane during the time interval $t = 0$ to $t = \tau$.

The function describing the point value of the flux $Q(t)$ has been defined in equation (11).

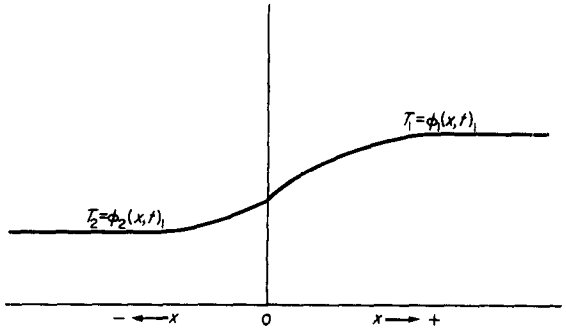

FIG. 2.

The time averaged value of the flux for a cycle with a period τ is given below:

$$\begin{aligned} Q_\tau = \frac{k_1}{\tau} \sum \int_0^{t_1} \frac{\partial}{\partial x}_{x=0} [\phi_1(x, t)_1]\, \mathrm{d}x \\ + \int_0^{t_2-t_1} \frac{\partial}{\partial x}_{x=0} [\phi_1(x, t)_2]\, \mathrm{d}x \ldots \\ + \int_0^{\tau-nt_1-mt_2} \frac{\partial}{\partial x}_{x=0} [\phi_1(x, t)_{m+n}]\, \mathrm{d}x \quad (21) \end{aligned}$$

i.e. Q_τ is the sum of integral mean fluxes between singularities. An expression similar to (21) can be derived in terms of $\phi_2(x, t)$.

Equation (19) is subject to the restriction contained in (15). The same technique can be applied for evaluating Q_τ for the more general case (i.e. for an arbitrary value of the t_1/t_2 ratio) although the resulting expressions would be considerably more complex.

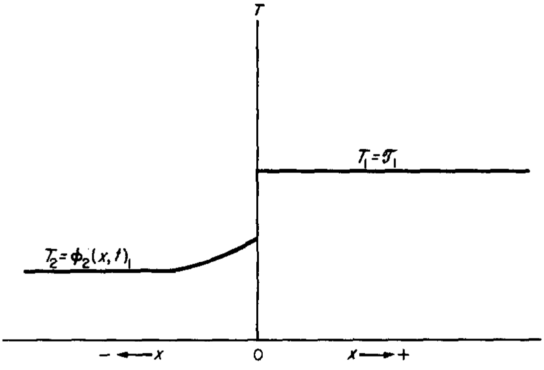

FIG. 3.

EVALUATION OF THE FUNCTIONS $\phi(x, t)$

In order to compute values for Q_τ the functions $\phi_1(x, t)_n$ and $\phi_2(x, t)_m$ must be determined. This can be done, at least for a simple case, by solving equations (1) and (2) for the appropriate boundary conditions.

For the $0 < t < t_1$ region the results are available from work on the bubble stirred interface [1].

i.e.
$$Q_1(t) = \frac{k_1}{\sqrt{(\pi\kappa_1 t)}} \times \frac{k_2\kappa_2^{-\frac{1}{2}}}{k_1\kappa_1^{-\frac{1}{2}} + k_2\kappa_2^{-\frac{1}{2}}} (\mathscr{T}_1 - \mathscr{T}_2) \quad (22)$$

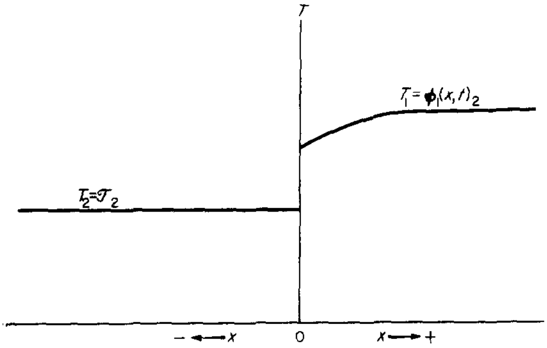

FIG. 4.

for the period $t_1 < t < t_2$ equations (1) and (2) have to be solved with the following boundary conditions:

$$T_1 = \mathscr{T}_2, \quad \text{at } x = 0 \quad t > 0 \tag{3}$$

$$k_1 \frac{\partial T_1}{\partial x} = k_2 \frac{\partial T_2}{\partial x}, \quad \text{at } x = 0 \quad t > 0 \tag{4}$$

$$T_1 = \mathscr{T}_1, \qquad \text{at } t = 0$$

and $T_2 = \mathscr{T}_2$

$$+ (\mathscr{T}_1 - \mathscr{T}_2) \frac{k_1\kappa_1^{-\frac{1}{2}}}{k_2\kappa_2^{-\frac{1}{2}} + k_1\kappa_1^{-\frac{1}{2}}} \operatorname{erfc} \frac{|x|}{2\sqrt{(\kappa_2 t_1)}}. \tag{23}$$

The solution of equations (1) and (2) with the above boundary conditions can be obtained for the flux [i.e. $k(\partial T_1/\partial x)_{x=0}$] by using Laplace transform. Details of the calculations are given in the Appendix. After some computation we have:

$$Q_2(t) = k_1(\mathscr{T}_1 - \mathscr{T}_2) \left[\frac{1}{\dfrac{1}{k_2\kappa_2^{-\frac{1}{2}}} + \dfrac{1}{k_1\kappa_1^{-\frac{1}{2}}}} \times \right.$$

$$\times \frac{1}{\sqrt{(\pi t)}} - \left(\frac{1}{\dfrac{1}{k_2\kappa_2^{-\frac{1}{2}}} + \dfrac{1}{k_1\kappa_1^{-\frac{1}{2}}} + \dfrac{k_2\kappa_2^{-\frac{1}{2}}}{k_1^2\kappa_1^{-1}}} \times \right.$$

$$\left.\left. \times \left\{ \frac{1}{\sqrt{(\pi t)}} - \frac{1}{\sqrt{[\pi(t+t_1)]}} \right\} \right) \right]. \tag{24}$$

Thus the mean value of the flux over the interval $(t_2 - t_1)$ is given as:

$$Q_{(t_2-t_1)} = \frac{1}{t_2 - t_1} \int_0^{t_2-t_1} Q_{(2)}(t)\,\mathrm{d}t$$

and the mean value of the flux over the interval $t_2 = t_1 + (t_2 - t_1)$ is:

$$Q_{t_2} = \frac{1}{t_2} \left[\int_0^{t_1} Q_1(t)\,\mathrm{d}t + \int_0^{(t_2-t_1)} Q_2(t)\,\mathrm{d}t \right]$$

$$= \frac{2(\mathscr{T}_1 - \mathscr{T}_2)}{t_2\sqrt{\pi}} \left(\frac{1}{\dfrac{1}{k_1\kappa_1^{-\frac{1}{2}}} + \dfrac{1}{k_2\kappa_2^{-\frac{1}{2}}}} [\sqrt{(t_1)} \right.$$

$$+ \sqrt{(t_2 - t_1)}] - \left\{ \frac{1}{\dfrac{1}{k_2\kappa_2^{-\frac{1}{2}}} + \dfrac{2}{k_1\kappa_1^{-\frac{1}{2}}} + \dfrac{k_2\kappa_2^{-\frac{1}{2}}}{k_1^2\kappa_1^{-1}}} \times \right.$$

$$\left.\left. \times \; [\sqrt{(t_1)} + \sqrt{(t_2 - t_1)} - \sqrt{(t_2)}] \right\} \right). \tag{25}$$

It can be seen that if $t_1 = t_2$ the expression in the {} brackets is zero and equation (23) reduces to that derived for the bubble stirred interface.

If we consider the simplest case* of nonequal surface renewal times, namely $t_2 = 2t_1$, then equation (23) gives the value for the mean flux over the complete cycle:

$$\tau = t_1 + (t_2 - t_1).$$

Thus we have:

$$Q = \frac{4\Delta T}{\sqrt{(2\pi t_2)}} \left[\frac{1}{\dfrac{1}{k_1\kappa_2^{-\frac{1}{2}}} + \dfrac{1}{k_2\kappa_2^{-\frac{1}{2}}}} \right.$$

$$\left. - \frac{1}{\dfrac{1}{k_2\kappa_2^{-\frac{1}{2}}} + \dfrac{1}{k_1\kappa_1^{-\frac{1}{2}}} + \dfrac{k_2\kappa_2^{-\frac{1}{2}}}{k_1^2\kappa_1^{-\frac{1}{2}}}} \left(\frac{2 - \sqrt{2}}{2} \right) \right]. \tag{26}$$

It is seen that the overall resistance to transfer can no longer be described as "composed of two linearly additive terms each containing parameters relating to only one of the phases".

The simplest schematic representation would be:

> overall resistance = resistance in one phase + resistance in the other phase + an interaction parameter.

---

* It is shown in the Appendix that analytical treatment of more complex cases would be extremely difficult; and numerical methods would be required to obtain a solution.

It can also be seen that if $k_1\kappa_1^{-\frac{1}{2}} \gg k_2\kappa_2^{-\frac{1}{2}}$ or $k_2\kappa_2^{-\frac{1}{2}} \gg k_1\kappa_1^{-\frac{1}{2}}$ the interaction parameter vanishes and the overall transfer rate can be expressed in terms of properties relating to phase 2 or phase 1, respectively.

DISCUSSION

A model has been proposed to describe the transfer across the interface of two independently stirred immiscible liquids. Uniform bulk properties were assumed, and the effect of stirring was considered to renew the surface of phase 1 at time intervals $t_1, 2t_1 \ldots\ldots nt_1$ and that of phase 2 at intervals $t_2, 2t_2 \ldots\ldots nt_2$. Transfer was considered to take place by unsteady state conduction during the time interval between the singularities (i.e. surface renewal of phase 1 or 2).

A scheme was suggested for evaluating a time averaged value of the flux crossing the interface during a cycle, a cycle being defined as the time interval between two successive points when the surface renewal times in the two phases coincide. In order to obtain the temperature distribution in the two phases and values for the flux, differential equations (1) and (2) had to be solved for successive time intervals between singularities; the boundary conditions relating to continuity, i.e. (3) and (4) remain the same for each time interval, the expressions giving the temperature distribution (i.e. the boundary condition relating to $t = 0$, $t = t_1$, $t = t_2$, etc.) are obtained from the solution for the previous time interval.

Thus at $t = 0$ we have $T_1 = \mathscr{T}_1$ (i.e. constant) and $T_2 = \mathscr{T}_2$ (i.e. constant). At $t = t_{1(-)}$ we have

$$T_1 = \phi_1(x, t) \text{ and } T_2 = \phi_2(x, t)_1 \qquad (15)$$

where equations (13) are naturally the solutions of equations (1) and (2) under the above boundary conditions. And at $t = t_{1(+)}$ we have $T_1 = \mathscr{T}_1$ and $T_2 = \phi_2(x, t)_1$ which are the new boundary conditions for equations (1) and (2) valid during the $t_2 - t_1$ time interval.

A similar procedure can be adopted for successive time intervals.

With regard to the actual calculations, functions $\phi_1(x, t)_1$ and $\phi_2(x, t)_1$ are available from previous work, thus the flux for the $t_1 - 0$ time interval is readily evaluated.

It is more difficult to obtain a solution for the interval $t_2 - t_1$, because of the complex boundary conditions. [$\phi_2(x, t)_1$ is an error function.] As shown in the Appendix, Laplace transform was used, and although the expression for the temperature distribution is not readily inverted the inverse transform describing the flux can be found in tables. Thus it was possible to evaluate the time averaged value of the flux for a cycle involving the simplest case of non-equal surface renewal times, i.e. when $t_2 = 2t_1$. It appears that an analytical solution for a more complex case would be rather difficult to obtain; such cases would be best solved using numerical methods.

However, interesting conclusions can be drawn from the analytical expression derived for this simplest case of non-equal surface renewal times. The overall resistance to transfer can no longer be described as "composed of two linearly additive terms each containing parameters relating to only one of the phases".

There exists an interaction parameter which contains factors relating to the physical properties of both phases.

It has also been shown that the physical properties of one phase can be such that the overall rate of the process is limited by that occurring in one phase only. In this case the interaction parameter vanishes and the resultant expression for the transfer rate reduces to an expression analogous to that given by Higbie [3] or Danckwerts [4].

Previous theories when describing interphase transfer postulated constant temperature or concentration at the interface. The model proposed in this paper does not contain this arbitrary restriction. It can be seen in the Appendix that for the nonequal surface renewal times in the two phases the temperature at $x = 0$ is no longer constant, but is a function of time. The time dependence of the interface temperature is likely reason for the nonadditivity of resistances.

The general difficulty experienced by many investigators when attempting to correlate mass-transfer results obtained for different systems may well be associated with the fact that the possible effects of an interaction parameter had not been taken into account.

It is of interest to note that liquid–liquid extraction results using a wide variety of materials were satisfactorily correlated by Lewis [5, 6] who employed an expression which involved three terms, one relating to one phase, the second to the other, and the third being essentially an interaction parameter.

This poses the question whether similar correlations should not be necessary for the accurate description of turbulent heat exchange between two fluids, when the "individual coefficients" are comparable in magnitude. It would appear that only a carefully conducted experimental investigation could give a definite answer.

REFERENCES

1. J. Szekely, Mathematical model for transfer at the bubble stirred interface of two immiscible liquids, *Int. J. Heat Mass Transfer*, **6**, 417 (1963).
2. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, p. 87. Oxford University Press (1959).
3. R. Higbie, The rate of absorption of a pure gas into a still liquid during short periods of exposure, *Trans. AIChE.* **31**, 365 (1935).
4. P. V. Danckwerts, Significance of liquid film coefficients in gas absorption, *Industr. Engng. Chem.* **43**, 1460 (1951).
5. J. B. Lewis, The mechanism of mass transfer of solutes across liquid–liquid interfaces, *Chem. Engng Sci.* **3**, 248 (1954).
6. J. B. Lewis, The mechanism of mass transfer of solutes across liquid–liquid interfaces, *Chem. Engng. Sci.* **3**, 260 (1954).
7. T. K. Sherwood, H. S. Mickley and C. E. Reed, *Applied Mathematics in Chemical Engineering*, p. 318. McGraw-Hill, New York (1957).

APPENDIX

To solve

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\kappa_1}\frac{\partial T_1}{\partial t}, \qquad x > 0 \qquad (1\text{A})$$

and

$$\frac{\partial^2 T_2}{\partial x^2} = \frac{1}{\kappa_2}\frac{\partial T_2}{\partial t}, \qquad x < 0 \qquad (2\text{A})$$

with the following boundary conditions:

$$T_1 = \mathscr{T}_2 \qquad \text{at } x = 0, \quad t > 0 \qquad (3\text{A})$$

$$k_1\frac{\partial T_1}{\partial x} = k_2\frac{\partial T_2}{\partial x} \quad \text{at } x = 0, \ t > 0 \qquad (4\text{A})$$

and

$$T_1 = \mathscr{T}_1 \quad \text{at } t = 0 \qquad (5\text{A})$$

$$T_2 = \mathscr{T}_2 + \frac{k_1\kappa_1^{-\frac{1}{2}}}{k_1\kappa_1^{-\frac{1}{2}} + k_2\kappa_2^{-\frac{1}{2}}}(\mathscr{T}_1 - \mathscr{T}_2)\,\mathrm{erfc}\,\frac{|x|}{2\sqrt{(\kappa_2 t_1)}} \qquad (6\text{A})$$

define:

$$M = \frac{k_1\kappa_1^{-\frac{1}{2}}\,\kappa_2^{-1}}{k_1\kappa_1^{-\frac{1}{2}} + k_2\kappa_2^{-\frac{1}{2}}}(\mathscr{T}_1 - \mathscr{T}_2) \qquad (7\text{A})$$

and

$$\sqrt{N} = \frac{1}{2\sqrt{(\kappa_2 t_1)}}. \qquad (8\text{A})$$

Apply Laplace transforms to equations (1A) and (2A):

$$\frac{\mathrm{d}^2\bar{T}_1}{\mathrm{d}x^2} - q_1^2\bar{T}_1 + \frac{\mathscr{T}_1}{\kappa_1} = 0 \qquad (9\text{A})$$

and

$$\frac{\mathrm{d}^2\bar{T}_2}{\mathrm{d}x^2} - q_2^2\bar{T}_2 + M\ \mathrm{erfc}\,\frac{|x|}{\sqrt{N}} + \frac{\mathscr{T}_2}{\kappa_2} = 0 \qquad (10\text{A})$$

where

$$q^2 \equiv \frac{p}{\kappa}: \qquad (11\text{A})$$

Equations (3A) and (4A) transform to give:

$$\bar{T}_1 = \bar{T}_2, \quad \text{at } x = 0 \qquad (12\text{A})$$

and

$$k_1\frac{\mathrm{d}\bar{T}_1}{\mathrm{d}x} = k_2\frac{\mathrm{d}\bar{T}_2}{\mathrm{d}x} \quad \text{at } x = 0. \qquad (13\text{A})$$

The solution of equation (9A) is given as:

$$\bar{T}_1 = C_1\,e^{-q_1 x} + C_2\,e^{q_1 x} + A_1(x) \qquad (14\text{A})$$

where C_1 and C_2 are constants and $A(x)$ is a particular integral.

As $\bar{T}_1$ is finite when $x \to \infty$, $C_2 = 0$; furthermore it can be shown that

$$A_1(x) = \frac{\mathscr{T}_1}{\kappa_1 q_1^2}. \qquad (15\text{A})$$

Therefore the solution of (9A) is:

$$\bar{T}_1 = C_1\,e^{-q_1 x} + \frac{\mathscr{T}_1}{\kappa_1 q_1^2}. \qquad (16\text{A})$$

The solution of (10A) is given as:

$$\bar{T}_2 = C_3\, e^{-q_1 x} + A_2(x). \tag{17A}$$

The particular integral $A_2(x)$ can be determined as follows:

$$A_2(x) = \frac{1}{(D + q_2)(D - q_2)}\left[\left(M \operatorname{erfc}\frac{|x|}{\sqrt{N}}\right) - \frac{\mathscr{T}_2}{\kappa_2}\right] \tag{18A}$$

where D is the differential operator.

Resolving into partial fractions we have:

$$A_2(x) = -\frac{1}{2q_2}\, e^{-q_2 x}\int e^{qx} M \operatorname{erfc}\frac{|x|}{\sqrt{N}}\,\mathrm{d}x + \frac{1}{2q_2}\, e^{q_2 x}\int e^{-q_2 x} M \operatorname{erfc}\frac{|x|}{\sqrt{N}}\,\mathrm{d}x + \frac{\mathscr{T}_2}{\kappa_2 q_2^2}\,. \tag{19A}$$

The integrals can be evaluated by integration in parts. After some computation we have:

$$A_2(x) = \frac{\mathscr{T}_2}{p} - \frac{M}{q_2^2}\left\{\operatorname{erfc}\frac{|x|}{\sqrt{N}} - \exp\,(q_2^2 N/4)\left[\tfrac{1}{2}\, e^{-q_2 x}\operatorname{erfc}\left(\frac{|x|}{\sqrt{N}} - \frac{q\sqrt{N}}{2}\right) + \tfrac{1}{2}\, e^{qx}\operatorname{erfc}\left(\frac{|x|}{\sqrt{N}} + \frac{q\sqrt{N}}{2}\right)\right]\right\}. \tag{20A}$$

Thus the solution of equation (10A) is given as:

$$\bar{T}_2 = \frac{\mathscr{T}_2}{p} + C_3\, e^{-q_2 x}\left\{\frac{M}{q_2^2}\operatorname{erfc}\frac{|x|}{\sqrt{N}} - \exp\,(q_2^2 N/2)\left[\tfrac{1}{2}\, e^{-q_2 x}\operatorname{erfc}\left(\frac{|x|}{\sqrt{N}} - \frac{q_2\sqrt{N}}{2}\right) + \operatorname{erfc}\left(\frac{|x|}{\sqrt{N}} + \frac{q_2\sqrt{N}}{2}\right)\times\tfrac{1}{2}\, e^{q_2 x}\right]\right\}. \tag{21A}$$

Substitution of the boundary conditions contained in equations (12A) and (13A) into equations (16A) and (21A) facilitates evaluation of constants C_1 and C_3.

Thus finally we have:

$$C_1 = \frac{\dfrac{M}{q_2^2}\left[\exp\,(q_2^2 N/4)\operatorname{erfc}\dfrac{q_2\sqrt{N}}{2} - 1\right] - \dfrac{\mathscr{T}_1 - \mathscr{T}_2}{p}}{1 + \dfrac{k_1 q_1}{k_2 q_2}} \tag{22A}$$

$$C_3 = \frac{\dfrac{M}{q_2^2}\left[\exp\,(q_2^2 N/4)\operatorname{erfc}\dfrac{q_2\sqrt{N}}{2} - 1\right] - \dfrac{\mathscr{T}_1 - \mathscr{T}_2}{p}}{1 + \dfrac{k_1 q_1}{k_2 q_2}} + \frac{\mathscr{T}_1 - \mathscr{T}_2}{p} + \frac{M}{q_2^2}. \tag{23A}$$

Values of C_1 and C_3 thus obtained can be substituted back into equations (16A) and (21A). To form the inverse transform of the resultant expression would require considerable manipulation and probably the use of contour integrals.

However, the expression giving the inverse transform of the flux: i.e.

$$\mathscr{L}^{-1}\left[k_1\left(\frac{\mathrm{d}\bar{T}_1}{\mathrm{d}x}\right)_{x=0}\right]$$

can be evaluated without difficulty [7]. Thus we have:

$$\bar{T}_1 = e^{-q_1 x}\,\frac{\dfrac{M}{q_2^2}\exp\,(q_2^2 N/4)\operatorname{erfc}\dfrac{q_2^2 N}{4} - \dfrac{M}{q_2^2} - \dfrac{\mathscr{T}_1 - \mathscr{T}_2}{p}}{1 + \dfrac{k_1 q_1}{k_2 q_2}} + \frac{\mathscr{T}_1 - \mathscr{T}_2}{p} \tag{24A}$$

and

$$\left(\frac{\partial \bar{T}_1}{\partial x}\right)_{x=0} = \frac{1}{1 + \dfrac{k_1 q_1}{k_2 q_2}}\left[\frac{M}{q_2^2}\, q_1 - \frac{M}{q_2^2}\, q_1 \exp\,(q_2^2 N/4)\times\operatorname{erfc}\frac{q_2\sqrt{N}}{2} + \frac{\mathscr{T}_1 - \mathscr{T}_2}{p\, q_1^{-1}}\right]. \tag{25A}$$

Noting that M, N and q were defined in equations (7A), (8A) and (11A) respectively, substituting back these original values and finally forming the inverse transforms using tables, we obtain:

$$k_1 \left(\frac{\partial T_1}{\partial x}\right)_{x=0} = k_1(\mathscr{T}_1 - \mathscr{T}_2)\frac{k_2\kappa_2^{-\frac{1}{2}}}{k_1\kappa_1^{-\frac{1}{2}} + k_2\kappa_2^{-\frac{1}{2}}}\left[\frac{1}{\sqrt{(\kappa_1\pi t)}} - \left(\frac{k_1\kappa_1^{-\frac{1}{2}}}{k_2\kappa_2^{-\frac{1}{2}} + k_1\kappa_1^{-\frac{1}{2}}} \times \right.\right.$$
$$\left.\left. \times \left(\frac{\kappa_2}{\kappa_1}\right)^{\frac{1}{2}}\left\{\frac{1}{\sqrt{(\pi\kappa_2 t)}} - \frac{1}{\sqrt{[\pi\kappa_2(t + t_1)]}}\right\}\right)\right]. \qquad (26\text{A})$$

It can be shown that, after some manipulation, equation (26A) is identical to equation (23) in the text.

Résumé—Cet article décrit un modèle pour les échanges à l'interface de deux liquides agités séparément. On considère que l'agitation est indépendante si les périodes de renouvellement des surfaces sont différentes pour les deux phases. On propose un schème pour déterminer une valeur, moyenne en fonction du temps, du flux qui traverse l'interface et on donne une solution analytique dans le cas très simple où les temps de renouvellement des surfaces sont inégaux, par exemple quand le temps de renouvellement d'une surface est deux fois celuide l'autre.

On montre que, même pour ce cas simple, la résistance totale à l'échange ne peut plus se décrire par "l'addition de deux termes linéaires contenant respectivement des paramètres se rapportant uniquement à chacune des phases". La représentation schematique la plus simple serait: résistance totale = résistance de l'une des phases + résistance dans l'autre phase + un paramètre d'interaction.

Zusammenfassung—Zur Beschreibung von Austauschvorgängen an der Trennfläche zweier unabhängig voneinander gerührten Flüssigkeiten wird ein Modell vorgeschlagen. Die unabhängige Durchrührung soll unterschiedliche Perioden der Oberflächenerneuerung in den beiden Phasen bewirken. Nach einem vorgeschlagenen Schema ist der zeitliche Mittelwert des Flusses quer zur Trennfläche zu bestimmen und eine analytische Lösung lässt sich für den einfachsten Fall ungleicher Oberflächenerneuerungszeit erhalten, d.h. wenn die Oberflächenerneuerungszeit in der einen Phase zweimal so gross ist wie in der anderen.

Es wird gezeigt, dass selbst für diesen einfachen Fall der Gesamtwiderstand des Austausches nicht mehr als "zusammengesetzt aus zwei linear additiven Ausdrücken, von denen jeder nur die auf eine Phase bezogenen Parameter enthält" beschrieben werden kann. Die einfachste schematische Wiedergabe wäre so: Gesamtwiderstand = Widerstand in einer Phase + Widerstand in der anderen Phase = Wechselwirkungsparameter.

Аннотация—Предложена модель описания переноса через поверхность раздела двух отдельно перемешиваемых жидкостей. Предполагалось, что в результате независимого перемешивания получаются различные периоды восстановления поверхности в двух фазах. Предложена схема расчёта среднего значения времени для потока, проходящего через поверхность раздела и получено аналитическое решение для простейшего случая неодинаковых периодов времени восстановления поверхности, т.е. когда восстановдение поверхности в одной фазе в два раза бодьше чем в другой.

Показано, что даже для этого простого случая, полное сопротивление переносу нельзя более описывать как «состоящее из двух аддитивных членов, каждый из которых содержит параметры, относящиеся только к одной из фаз». Простейшим схематическим представлением является: полное сопротивдение = сопротивлению в одной фазе + сопротивление в другой фазе + параметр взаимодействия.